Role of Spin in the Monopole Problem

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The scattering of an electric charge from a magnetic monopole is discussed in a way which explicitly incorporates conservation of angular momentum. The Dirac quantization condition for the physical charges is derived from the correspondence principle and the requirement of rotational invariance. The same discussion shows that the initial and final states of the scattering reaction contain an extra spin, which cannot be associated with either particle alone. In the classical nonrelativistic theory it is known that such an extra spin appears, and that it may be identified with the angular momentum of the electromagnetic field. A quantized version of this nonrelativistic spin theory is obtained and shown to be equivalent to the Dirac theory based on a singular vector potential. The spin approach gives an interesting perspective on the relativistic monopole problem. Among the standard S-matrix postulates, that of crossing symmetry must be modified or abandoned if a relativistic theory is to succeed.

I. INTRODUCTION

In 1931 Dirac[1] introduced the hypothesis of a new particle, the magnetic monopole. Such a particle, for which there is no experimental evidence,[2] would be a source of magnetic field corresponding to a point-charge source of electric field. He argued that if the new particle were to fit into conventional quantum mechanics, it must satisfy a quantization condition

$$eg/\hbar c=n/2\,, \tag{1.1}$$

where g is the charge of the monopole, e is the electric charge of any other particle, and n is some integer. It follows immediately from this condition that all electric charges are integer multiples of a smallest charge, if even one monopole exists. Thus, the existence of monopoles would "explain" the quantization of charge.

A semiclassical derivation of (1.1) was given by Wilson,[3] who noted that for a static system of a charge and a monopole there is an angular momentum in the electromagnetic field,

$$\mathbf{s}=(8\pi c)^{-1}\int d^3r\, \mathbf{r}\times(\mathbf{E}\times\mathbf{H})=(eg/c)\hat{n}\,, \tag{1.2}$$

where $\hat{n}$ is a unit vector from e to g. Quantizing $\mathbf{s}$ in units of $\frac{1}{2}\hbar$ recovers (1.1).

The relation of Wilson's argument to Dirac's has remained obscure. In this paper, I show that the spin approach is indeed relevant and permits much insight into the quantum problem. In Sec. II, we shall see that general principles of quantum mechanics require the presence of an extra spin in the initial and final states for scattering of a spinless monopole from a spinless charge. The same considerations also yield the Dirac quantization condition. Section III gives the nonrelativistic theory and demonstrates that the extra spin may be identified with the classical spin of Wilson. The relation to Dirac's formulation is made clear. Finally, in Sec. IV are found certain constraints on the undeveloped relativistic theory of charge-monopole scattering.

II. THE CORRESPONDENCE PRINCIPLE AND THE S MATRIX

Let us examine the consequences of some minimal requirements on a quantum theory of monopoles. For large impact parameter $\mathbf{b}$, we may calculate the scattering angle in the impulse approximation. The force is

$$\mathbf{F}(\mathbf{r},\mathbf{v})=(e\mathbf{v}/c)\times g\hat{r}/r^2\,,$$

giving

$$\Delta p\approx\int_{-\infty}^{\infty}dt\,F(\mathbf{v}_0t+\mathbf{b},\mathbf{v}_0)$$
$$=(2eg/bc)\hat{v}_0\times\hat{b}\,. \tag{2.1}$$

If we write the final momentum $\mathbf{p}'$ in polar coordinates (θ,φ) with the initial momentum $\mathbf{p}=\mu\mathbf{v}_0$ as axis, we may invert (2.1) to read

$$\mathbf{b}\approx-(2eg/cp\theta)\hat{\varphi},\quad(\theta\ll1) \tag{2.2}$$

where $\hat{\varphi}$ is a unit vector in the direction of increasing azimuthal angle φ in a right-handed coordinate system.

The quantum-mechanical scattering of a spinless charge from a spinless monopole for an incoming plane wave must lead to an asymptotic scattered wave of the usual form,

$$\psi_{\rm in}=e^{ikz}\,,$$
$$\psi_{\rm scatt}\sim(e^{ikr}/r)f(\theta,\varphi)\,, \tag{2.3}$$
$$p=\hbar k\,.$$

The logarithmic radial factors familiar in the Coulomb problem are not expected because the force here falls

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[1] P. A. M. Dirac, Proc. Roy. Soc. (London) A133, 60 (1931); Phys. Rev. 74, 817 (1948).
[2] W. V. R. Malkus, Phys. Rev. 83, 899 (1951); H. C. Fitz, W. B. Good, J. L. Kassner, and A. E. Ruark, *ibid.* 111, 1406 (1958); H. Bradner and W. M. Isbell, *ibid.* 114, 603 (1959); E. M. Purcell, G. B. Collins, T. Fujii, J. Hornbostel, and F. Turkot, *ibid.* 129, 2326 (1963); E. Amaldi, G. Baroni, H. Bradner, L. Hoffman, A. Manfredini, and G. Vanderhaeghe, Nuovo Cimento 28, 773 (1963); E. Goto, H. H. Kolm, and K. W. Ford, Phys. Rev. 132, 387 (1963).
[3] M. N. Saha, Indian J. Phys. 10, 145 (1936); Phys. Rev. 75, 1968 (1949). H. A. Wilson, *ibid.* 75, 309 (1949).

off as r^{-3} when the scattered particle is far away compared to its impact parameter.

If an incoming wave packet is constructed, application of the method of steepest descent leads to a relation[4] between the impact parameter of the wave packet and the angle of scattering

$$\mathbf{b}=(\nabla_{\mathbf{k}}\alpha)_{\perp}, \tag{2.4}$$

where $f=|f|e^{i\alpha}$, $\nabla_{\mathbf{k}}$ represents the gradient with respect to the initial wave number, and $\perp$ symbolizes the part of the resulting vector perpendicular to $\mathbf{k}$. The parallel part contributes to a "time delay"[5] which does not concern us here. We may rewrite (2.4) in terms of θ and φ as

$$\mathbf{b}=(\hat{k}'-\hat{k}'\cdot\hat{k}\hat{k})(1/k)\partial\alpha/\partial\cos\theta - \hat{\varphi}(1/k\sin\theta)\partial\alpha/\partial\varphi. \tag{2.5}$$

Comparing with (2.2), we obtain for $\theta\ll 1$:

$$\alpha\approx(2eg/\hbar c)\varphi. \tag{2.6}$$

To appreciate the significance of this strange azimuthal dependence of the scattering amplitude we must turn to the well-known implications of rotational invariance in quantum mechanics. Consider a transition from initial to final state described by an amplitude $\langle o|T|i\rangle$. If the resulting transition probability is to be invariant under rotations, we must require

$$|\langle o|T|i\rangle|=|\langle Ro|T|Ri\rangle|=|\langle o|R^{-1}TR|i\rangle|. \tag{2.7}$$

This implies

$$\langle o|R^{-1}TR|i\rangle=e^{i\Phi(R)}\langle o|T|i\rangle. \tag{2.8}$$

The phase $\Phi(R)$ may not depend on $|o\rangle$ or $|i\rangle$ if (2.7) is to hold for arbitrary wave packets. Writing $R=R_2R_1$, we have, by successive application of (2.8),

$$\Phi(R_2R_1)=\Phi(R_1)+\Phi(R_2). \tag{2.9}$$

Any rotation may be written in the form

$$R_1=R_2R_1^{-1}R_2^{-1},$$

and so Φ is identically zero, or

$$\langle o|R^{-1}TR|i\rangle=\langle o|T|i\rangle,$$
$$R^{-1}TR=T. \tag{2.10}$$

If we call $\mathbf{J}$ the generator of rotations on $|i\rangle$ or $|o\rangle$ (assumed the same for both), then $\mathbf{J}$ is conserved,

$$[\mathbf{J},T]=0, \tag{2.11}$$

and we may think of $\mathbf{J}$ as a conserved total angular momentum.

This brings out an important distinction between classical and quantum theory. In a classical theory with an arbitrary force law there is no reason to expect a conserved total angular momentum, even if energy and linear momentum are conserved. In quantum theory, general invariance requirements, combined with the linearity of the theory, guarantee the existence of a $\mathbf{J}$ which commutes with the S matrix. As a result, some classical theories might have no quantum analog, and others might have a quantum analog only for a restricted class of parameters, in order to permit the existence of a conserved quantized total angular momentum. The latter case obtains in the problem at hand. The usual ground for conservation of angular momentum in classical theory is the existence of a rotationally invariant Hamiltonian. We shall see that there is such a Hamiltonian for this problem, but that it has a very peculiar nature. Quantization of the angular momentum appearing in the Hamiltonian leads to the Dirac condition.

Returning to (2.6), suppose for the moment that the initial and final states are described, as is usual, only by the momenta of the colliding particles. Then (2.10) implies that $\langle o|T|i\rangle$ depends only on $\mathbf{k}\cdot\mathbf{k}'$ and k^2, or equally well, on E and θ, where E is the energy. This would contradict (2.6), which shows that $f=\langle o|T|i\rangle$ depends on φ. Thus, an extra factor χ must be introduced, which depends on some parameter other than the momenta. If we call $\mathbf{L}$ the generator of rotations on the momenta, i.e., the orbital angular momentum, and $\mathbf{s}$ the generator of rotations on the new parameter in χ, then $\mathbf{J}=\mathbf{L}+\mathbf{s}$ may be the conserved angular momentum in (2.10). Matrix elements of T between states with different z components of $\mathbf{s}$, m and m', will have a φ dependence $\alpha=(m-m')\varphi$. Since $(m-m')$ is an integer, we obtain the Dirac quantization condition that $2eg/\hbar c$ is an integer. This argument, then, gives a necessary condition for the existence of monopoles, but does not show that the condition can be satisfied.

One might think that the argument above is unnecessary, that continuity of the wave function requires the Dirac quantization condition in (2.6). As we shall see in Sec. IIIC, this is not the case.

Note that the spin $\mathbf{s}$ may not be identified as the intrinsic spin of the new particle, the monopole. If the monopole had a spin of magnitude S, then the maximum value of $|m-m'|$ would be $2S$, and (2.6) would imply $eg/\hbar c\leq S$. By using projectiles of arbitrarily large charge e, one could always violate this condition. Thus, $\mathbf{s}$ may not be attached to either particle alone, but depends on both.

We may summarize the assumptions and conclusions of this section thus.

If (1) the quantum-mechanical theory of charge-monopole interaction reproduces the large impact parameter, small-angle scattering implied by the classical Lorentz force law, and (2) the differential cross section is invariant under rotation of the initial and final momenta through the same angle, then: (1) the initial and final states may not be described merely by a product of wave functions for freely moving charge and monopole; an additional factor corresponding to an

[4] See, for example, M. Froissart, M. L. Goldberger, and K. M. Watson, Phys. Rev. **131**, 2820 (1963).

[5] E. P. Wigner, Phys. Rev. **98**, 145 (1955); M. L. Goldberger and K. M. Watson, *ibid.* **127**, 2284 (1962).

extra spin **s** is required; and (2) the product of charges must obey the Dirac condition.

In the following section, we shall find that the extra spin **s** may be identified with the classical-field angular momentum of Wilson, in a nonrelativistic theory. By treating **s** as an independent variable, we shall obtain a rotationally invariant Hamiltonian in a *post hoc* justification of the existence of a conserved total angular momentum.

III. NONRELATIVISTIC THEORY

A. The Classical Case

The equation of motion[6] of a charge e with mass μ in the field of a fixed monopole of charge g is

$$\mu\ddot{\mathbf{r}}=(eg/c)\mathbf{r}\times\dot{\mathbf{r}}/r^3. \tag{3.1}$$

It follows immediately that $v=|\dot{\mathbf{r}}|$ and $|\mathbf{L}|=|\mathbf{r}\times\mu\dot{\mathbf{r}}|$ are constants of the motion. Thus, in the revolving plane containing the two charges, with $\mathbf{L}$ as instantaneous normal, the charge moves in a straight line with uniform velocity whose magnitude is v. The distance of the line from the origin is the impact parameter b. This yields the relation

$$r=[(vt)^2+b^2]^{1/2}. \tag{3.2}$$

The problem is reduced to finding the motion of the revolving plane. To do this, recall the spin **s** introduced in Sec. I, $\mathbf{s}=(-eg/c)\hat{r}$. Taking $\mathbf{J}=\mathbf{L}+\mathbf{s}$, one may deduce from (3.1) and this definition that

$$d\mathbf{J}/dt=0,\quad d\mathbf{J}\cdot\mathbf{s}/dt=0,\quad d\mathbf{J}\cdot\mathbf{L}/dt=0. \tag{3.3}$$

All that remains is to compute the time dependence of ω, the azimuthal angle of **s** about **J**. With the help of (3.1)–(3.3) we obtain

$$\dot{\omega}=(L^2+s^2)^{1/2}/\mu r^2=(eg/\mu c\cos\psi)\{1/[(vt)^2+b^2]\},\quad \cot\psi=s/L=eg/\mu vbc, \tag{3.4}$$

leading to

$$\omega(\infty)=eg\pi/vb\mu c\cos\psi=\pi/\sin\psi. \tag{3.5}$$

Finally, the polar scattering angle θ and differential cross section are given by

$$\cos\theta=-\cos^2\psi+\sin^2\psi\cos(\pi/\sin\psi),$$
$$\frac{d\sigma}{d\Omega}=\frac{(eg/\mu vc)^2\sin\psi/\cos^4\Psi}{|2\sin\psi[1-\cos(\pi/\sin\psi)]-\pi\sin(\pi/\sin\psi)|}. \tag{3.6}$$

The motion is on the surface of a cone, with axis **J** and half-angle ψ. The cross section at small θ is like that for the Coulomb scattering from a fixed charge $e'=gv/c$. At large angles $d\sigma/d\Omega$ has integrable singularities, with a cumulation point at $\theta=\pi$, which occur because θ is a bounded oscillating function of b^{-1}. In geometrical terms, the charge spirals around the cone more and more often as b goes to zero, and can scatter through the same polar angle for several different trajectories.

Since there is a conserved total angular momentum, it should not be surprising that there is a Hamiltonian formulation of the problem. Consider

$$H=p_r^2/2\mu+(\mathbf{J}^2-s^2)/2\mu r^2. \tag{3.7}$$

Since **L** and **s** are perpendicular, this is precisely the kinetic energy of the charge. However, we may interpret H as a Hamiltonian, provided **s** is taken as an angular momentum with independent degrees of freedom, obeying the Poisson-bracket relation,

$$\{s_i,s_j\}=-\epsilon_{ijk}s_k. \tag{3.8}$$

The equations of motion become

$$\begin{aligned}
\dot{\mathbf{r}}&=\{H,\mathbf{r}\}=(1/\mu)[\mathbf{p}-(\mathbf{r}\times\mathbf{s})/r^2],\\
\dot{\mathbf{p}}&=\{H,\mathbf{p}\}=-(1/\mu r^2)[\mathbf{p}\times\mathbf{s}-2(\mathbf{p}\times\mathbf{s})\cdot\hat{r}\hat{r}],\\
\dot{\mathbf{s}}&=-d\mathbf{L}/dt=(\mathbf{L}\times\mathbf{s})/\mu r^2=(\mathbf{J}\times\mathbf{s})/\mu r^2,\\
d\hat{r}/dt&=(\mathbf{L}\times\hat{r})/\mu r^2+(\mathbf{s}\times\hat{r})/\mu r^2=(\mathbf{J}\times\hat{r})/\mu r^2.
\end{aligned} \tag{3.9}$$

It follows immediately that if the condition $\mathbf{s}=(-eg/c)\hat{r}$ is imposed at one time it will remain true, and all the unwanted terms in (3.9) will disappear.

In Part B, the Hamiltonian (3.7) will be used in the quantum problem to obtain a solution along the lines required by Sec. II.

B. Quantum Theory for the Spin Hamiltonian

For the quantum case, we interpret the momentum operator in (3.7) as $\mathbf{p}=-i\hbar\nabla$, and the spin **s** as a quantum-mechanical angular momentum $\hbar\mathbf{s}$ with the commutation relations $\mathbf{s}\times\mathbf{s}=i\mathbf{s}$. Since H is positive definite, there are no bound states,[7] and we need only determine the scattering solutions of $H\Psi=E\Psi$. First, note that

$$[\mathbf{s}\cdot\hat{r},H]=0. \tag{3.10}$$

This may be deduced from the fact that $\mathbf{J}=\mathbf{L}+\mathbf{s}$, and **L** generates rotations on $\hat{r}$ while **s** generates rotations on **s**. Therefore, **J** leaves $\mathbf{s}\cdot\hat{r}$ invariant; since H contains only $\mathbf{J}^2$ and operators on r, (3.10) follows. However, if the ratio of $\mathbf{L}\cdot\mathbf{s}$ to $\mathbf{L}^2$ in H were changed, (3.10) would be false.

Using (3.10), we may look for a solution which follows the classical analogy, obeying

$$\mathbf{s}\cdot\hat{r}\Psi=(-eg/\hbar c)\Psi \tag{3.11}$$

which entails the Dirac quantization condition. The general case is treated in the Appendix, and we may

[6] The classical trajectory has been treated, for example, by H. Poincaré, Compt. Rend. **123**, 530 (1896); M. Fierz, Helv. Phys. Acta **17**, 27 (1944); I. R. Lapidus and J. L. Pietenpol, Am. J. Phys. **28**, 17 (1960); G. Nadeau, *ibid.* **28**, 566 (1960). Note that this calculation applies equally well to the motion of a charge and a monopole of finite mass if **r** is taken as the relative coordinate, and μ, as the reduced mass.

[7] But see Ref. 23. The absence of bound states was noted by Dirac, Ref. 1.

confine ourselves here to the case $\mathbf{s}=(\hbar/2)\boldsymbol{\sigma}$, i.e., spin $\frac{1}{2}$. We take as incident wave

$$\Psi_{\text{inc}}=e^{ikz}\begin{pmatrix}1\\0\end{pmatrix}. \tag{3.12}$$

Since the Hamiltonian (3.7) leads to essentially constant phase shifts for arbitrarily high l, this incoming wave requires interpretation. The meaning is that wave packets constructed from the "plane-wave" solutions will take the form

$$\Psi(t)=e^{-iHt}\lim_{\tau\to-\infty} e^{iH\tau}e^{-iH_0\tau}\Psi_{\text{inc}}, \tag{3.13}$$

where H_0 is the kinetic energy alone. In practice this means that matching of ψ to ψ_{inc} is accomplished by evaluating Bessel functions from their asymptotic forms ($kr\gg l$) for arbitrarily high l, and matching coefficients of e^{-ikr}.

Partial wave expansion gives

$$\Psi_{\text{inc}}=\sum_{l=0}^{\infty}(2^l+1)i^l j_l(kr)P_l(\cos\theta)\chi_u, \tag{3.14}$$

in standard notation, with

$$\chi_u=\begin{pmatrix}1\\0\end{pmatrix}.$$

Converting the expansion to states of definite J gives

$$\Psi_{\text{inc}}=\sum_{J=\frac{1}{2}}^{\infty}\{(\lambda+1)i^\lambda[P_\lambda j_\lambda+iP_{\lambda+1}j_{\lambda+1}]\chi_u + i^{\lambda+1}\sin\theta\,\boldsymbol{\sigma}\cdot\hat{\varphi}[P_\lambda' j_\lambda-iP_{\lambda+1}'j_{\lambda+1}]\chi_u\}, \tag{3.15}$$
$$\lambda=J-\tfrac{1}{2}.$$

Examination of (3.7) shows that the true radial-wave function for a given J is $j_q(kr)$, with q given by

$$q+\tfrac{1}{2}=[(J+\tfrac{1}{2})^2-s^2]^{1/2}=[(J+\tfrac{1}{2})^2-\tfrac{1}{4}]^{1/2}. \tag{3.16}$$

Using the asymptotic form

$$j_\zeta(z)\underset{z\to\infty}{\longrightarrow}z^{-1}\sin(z-\zeta\pi/2), \tag{3.17}$$

we match incoming waves to obtain Ψ from Ψ_{inc}:

$$\begin{aligned}\Psi&=\sum_{J=\frac{1}{2}}^{\infty}(-1)^\lambda i^{-q}j_q[(\lambda+1)(P_\lambda-P_{\lambda+1}) + i\sin\theta\,\boldsymbol{\sigma}\cdot\hat{\varphi}(P_\lambda'+P_{\lambda+1}')]\chi_u\\ &=2\sin(\theta/2)\sum_{J=\frac{1}{2}}^{\infty}(-1)^\lambda i^{-q}j_q(P_\lambda'+P_{\lambda+1}')\times\exp[i(\boldsymbol{\sigma}/2)\cdot\hat{\varphi}\bar{\theta}]\chi_u,\end{aligned} \tag{3.18}$$
$$\bar{\theta}=\pi-\theta.$$

The last step follows from the identity

$$(\lambda+1)(P_\lambda-P_{\lambda+1})=2\sin^2(\theta/2)(P_\lambda'+P_{\lambda+1}'). \tag{3.19}$$

Thus, the exact solution is a spinor which always points in the inward radial direction. In fact, one may verify directly that (3.7) is diagonalized in the spin-$\frac{1}{2}$ space by transformation to the radius-based coordinates:

$$\begin{aligned}H'&=\exp[i(\boldsymbol{\sigma}/2)\cdot\hat{\varphi}\theta]H\exp[-i(\boldsymbol{\sigma}/2)\cdot\hat{\varphi}\theta]\\ &=\frac{-\hbar^2}{2\mu}\left\{\frac{1}{r}\partial_r{}^2r+\frac{1}{r^2}\left[\frac{1}{\sin\theta}\partial_\theta\sin\theta\partial_\theta - \frac{1}{\sin^2\theta}[-i\partial_\varphi+\tfrac{1}{2}\sigma_z(1-\cos\theta)]^2\right]\right\}.\end{aligned} \tag{3.20}$$

This Hamiltonian is identical with that of Dirac, (3.26), for the case $eg/\hbar c=\pm\frac{1}{2}$, provided σ_z is taken as $\sigma_z=\mp1$. This is what we should expect from the angular-momentum interpretation, since a positive product of charges corresponds to an inward radial $\mathbf{s}$.

To obtain the scattering amplitude, we look at the outgoing wave:

$$\begin{aligned}\Psi_{\text{scatt}}&=(e^{ikr}/r)\mathbf{F}=(e^{ikr}/ikr)\sin(\theta/2)\\ &\times\exp[i(\boldsymbol{\sigma}/2)\cdot\hat{\varphi}\bar{\theta}]\chi_u\sum_{J=\frac{1}{2}}^{\infty}(-1)^{\lambda-q}(P_\lambda'+P_{\lambda+1}').\end{aligned} \tag{3.21}$$

For small θ, only the asymptotic summand is significant, yielding

$$\begin{aligned}\mathbf{F}(\theta\ll1)&\approx-i\sin(\theta/2)\exp[i(\boldsymbol{\sigma}/2)\cdot\hat{\varphi}\bar{\theta}]\chi_u\times\sum_{\lambda=0}^{\infty}(-i)(d/d\cos\theta)(P_\lambda+P_{\lambda+1})\\ &=-\exp[i(\boldsymbol{\sigma}/2)\cdot\hat{\varphi}\bar{\theta}]\chi_u[eg(v/c)/2\mu v^2\sin^2(\theta/2)],\\ f(\theta\ll1,\varphi)&=\chi_d{}^\dagger\mathbf{F}\chi_u=-e^{i\varphi}|f|\\ &=-e^{2i(eg/\hbar c)\varphi}(d\sigma/d\Omega)^{1/2}.\end{aligned} \tag{3.22}$$

As one examines this solution, it is helpful to consider the asymptotic behavior of the wave function. At first glance, the "interaction term" $(2\mu r^2)^{-1}(\boldsymbol{\sigma}\cdot\mathbf{L}+\frac{1}{2})$ appears to fall inversely with r^2. However, for a plane wave e^{ikz}, the orbital angular momentum is $k\mathbf{r}\times\hat{z}$, and the interaction falls only as r^{-1} for directions perpendicular to $\hat{z}$. This agrees with the crudest classical reasoning since the Lorentz force falls as r^{-3} along a particle trajectory, but only as r^{-2} in orthogonal directions. If we look for an "asymptotic" ψ_a which satisfies the Schrödinger equation through terms of order $r^{-1}E\psi_a$, then e^{ikz} must be multiplied by a modifying factor. In the analogous Coulomb case, that factor is an imaginary power of $(r-z)$.[8] Here it turns out to be

$$\exp[-i(\boldsymbol{\sigma}/2)\cdot\hat{\varphi}\theta],$$

precisely the quantity required to transform H to H',

[8] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 116.

the Dirac Hamiltonian (3.20). By inspection, this new H produces no terms of order r^{-1} when acting on e^{ikz}. The initial condition that a plane wave come in from the $-z$ direction requires an additional factor so that ψ_a becomes $e^{i\varphi}e^{ikz}$. This azimuthal dependence may be checked against the explicit form (3.18), where it comes from the factor

$$\exp[i\pi(\boldsymbol{\sigma}/2)\cdot\hat{\varphi}].$$

Acting on a function $e^{i\varphi}\ g(r,\theta)$, H diverges for $\theta=0$. Thus, ψ_a must vanish on the positive z axis. In fact, it must go to zero as fast as θ. The result is that ψ_a is only a reasonable asymptotic form if it is evaluated outside a cone $\theta<\theta_0$, and correction terms become small for $kr\theta_0\gg 1$. More simply put, the excluded region is a cylinder of radius ρ about the positive z axis, with $k\rho\gg 1$. This is reasonable because particles with impact parameter $b\ll\rho$ are scattered significantly. The zero in ψ at $\theta=0$ may be verified explicitly from (3.18). For fixed r and large $q\approx J$, the spherical Bessel function $j_q(kr)$ takes the form $j\sim(e/\sqrt{2})(2J+1)^{-1}[ekr/(2J+1)]^J$.[9] A crude bound on $\sin(\theta/2)(P_\lambda'+P_{\lambda+1}')=(2J+1)d_{-1/2,1/2}$[10] is found from $|d^J|<CJ\theta 2^{2J}$,[11] where C is a constant. Substituting these expressions in (3.18) gives the behavior near $\theta=0$, $|\psi|<C'\theta$ for fixed r.

This statement seems inconsistent with the divergent expression for f, Eq. (3.22). However, that expression was obtained with the assumption that all Bessel functions could be evaluated in the large-r asymptotic region. That assumption is only valid for $J\ll kr$, and therefore the series for f will really be cut off at $J\approx kr$. To see the effect of such a cutoff, let us look at the generating function

$$(1+z^2-2xz)^{-1/2}=\sum_{\lambda=0}^{\infty} z^\lambda P_\lambda(x).$$

The truncated series for f may be estimated as the derivative with respect to $\theta=\cos^{-1}x$ of the generating function, with z taken as, say, $(10)^{-1/kr}$. This means that terms with $\lambda\approx kr$ will be suppressed by a factor $\frac{1}{10}$. The resulting condition for f to agree with (3.22) is $kr\theta^2\gg 1$. This is even stronger than the condition $kr\theta\gg 1$ that a plane wave be an asymptotic solution, but that is not surprising, since (3.22) diverges at $\theta=0$, while a plane wave does not. The exclusion of $kr\theta^2\lesssim 1$ from the domain of expression (3.22) for f may be explained intuitively as follows. The differential cross section, and thence the scattering amplitude, diverge at small θ. This divergence corresponds to trajectories far away from the z axis, but, if θ goes to zero for fixed r, the trajectory touches the z axis and should not lead to a divergence. It is worth noting that corresponding constraints apply to the interpretation of conventional asymptotic expressions for the Coulomb scattering wave function.[8]

We have obtained a nonrelativistic theory consistent with the requirements of Sec. II. In Part C, we shall compare this theory with the Banderet[12] solution of Dirac's equation for charge-monopole scattering.

C. Quantum Theory with a Vector Potential

Dirac's discussion[1] of the quantum problem depends on the introduction of a singular vector potential to represent the field of a fixed monopole,

$$\mathbf{A}=g(\hat{\varphi}/r)\tan(\theta/2). \qquad (3.23)$$

This potential obeys the relations

$$\nabla\times\mathbf{A}=(g\hat{r}/r^2),\ (\theta\neq\pi)$$

$$\int_C \mathbf{A}\cdot d\mathbf{x}=-4\pi g, \qquad (3.24)$$

where C is a small circuit about the line $\theta=\pi$. Thus, $\mathbf{A}$ corresponds not to an isolated monopole charge, but rather to a magnetic flux line extending from zero to infinity along the negative z axis, a long thin dipole. The charge-monopole interaction is introduced through the usual substitution $\nabla\rightarrow\nabla-(ie/\hbar c)\mathbf{A}$. Since the z axis holds no special significance, one should be able to turn the dipole line in any way without changing the physical results. On making such a change, one finds that the wave function ψ' which obeys the same equations as ψ, but in terms of the new potential $\mathbf{A}'$, is

$$\psi'(\mathbf{x})=\{\exp[(ie/\hbar c)\textstyle\int^{\mathbf{x}}d\mathbf{x}'\cdot(\mathbf{A}'-\mathbf{A})]\}\psi(\mathbf{x}). \qquad (3.25)$$

We require the phase factor to be well defined. For paths which do not enclose the singular lines of $\mathbf{A}$ or $\mathbf{A}'$, the phase is well defined, since $\nabla\times(\mathbf{A}'-\mathbf{A})=0$. However, paths which encircle either singular line once lead to a phase change $\Delta\Phi=\pm 4\pi(eg/\hbar c)$. This must be a multiple of 2π; hence, the Dirac condition follows.

This gauge or dipole line invariance has more than formal significance. As pointed out most forcefully by Aharonov and Bohm,[13] a flux line which is *not* penetrated by the charge may still affect scattering. In particular, the diffraction pattern of an electron wave passing on either side of a flux line shifts by one fringe each time the enclosed flux changes by $2\pi\hbar c/e$.[14] Thus, the dipole

[9] The spherical Bessel function is related to the ordinary Bessel function by $j_\nu(z)=(\pi/2z)^{1/2}J_{\nu+1/2}\ (z)$. For the asymptotic form see M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards Applied Mathematics Series-55 (U. S. Government Printing Office, Washington, D.C., 1965), p. 365.

[10] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.), **7**, 426 (1959).

[11] See M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957), p. 53.

[12] P. P. Banderet, Helv. Phys. Acta **19**, 503 (1946).

[13] W. E. Ehrenburg and R. E. Siday, Proc. Phys. Soc. (London) **B62**, 8 (1949); Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959); **123**, 1511 (1961); W. H. Furry and N. F. Ramsey, *ibid.* **118**, 623 (1960).

[14] This has been verified experimentally by G. Möllenstedt and W. Bayh, Naturwiss. **49**, 81 (1962). An analog experiment using the Josephson effect in superconductors was done by R. C. Jaklevic, J. J. Lambe, A. H. Silver, and J. E. Mercereau, Phys. Rev. Letters **12**, 275 (1964).

line *would* have an observable effect without the Dirac condition on the flux $4\pi g$. There is, then, a close connection between the rotational invariance argument of Sec. II and the gauge-invariance argument here. Note that if we omit these invariance conditions and allow a given line to be a line of observable singularity, then the wave function may be discontinuous or undefined for paths encircling this line. The resulting Hamiltonian would be an acceptable operator, but it would not describe the scattering of a charge from a magnetic monopole. To adopt the language of Dirac,[1] there would be a "string" attached to the monopole, and the orientation of the incoming beam relative to the string would influence the scattering, thus violating rotational invariance. A specification of "monopole" position would require determination of a line, not a point. There would be an uncountable number of degrees of freedom, and the "monopole" would not be a particle in the usual sense. Thus, without the quantization condition, Dirac's H is appropriate for scattering from an infinite dipole with one end at the origin, but *not* for scattering from a monopole particle. This discussion explains why, in Sec. II, continuity in φ could not be required *ab initio* but only as a consequence of rotational invariance.

Let us examine the solutions of Dirac's equation[1] when the quantum condition is obeyed:

$$\hbar^2k^2/2\mu = H = (1/2\mu)(\mathbf{p}-(e/c)\mathbf{A})^2$$
$$= -\frac{\hbar^2}{2\mu}\left\{\frac{1}{r}\partial_r{}^2 r + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\partial_\theta \sin\theta\partial_\theta - (-i\partial_\varphi - s(1-\cos\theta))^2\right]\right\},$$
$$s = eg/\hbar c. \tag{3.26}$$

For $s=\frac{1}{2}$, this agrees with (3.18). The solutions of (3.26) are obtained by separation of variables as[15]

$$\Psi_{Jm} = j_q(kr)e^{im\varphi}d_{s-m,s}{}^J(\theta),$$
$$q+\tfrac{1}{2} = [(J+\tfrac{1}{2})^2 - s^2]^{1/2}, \quad J \geqq |s|, \quad |s-m| \tag{3.27}$$

where j_q, again, is a spherical Bessel function, and d^J is a rotation function or helicity amplitude.

Banderet[12] obtained a scattering solution for a wave going in the *negative* z direction by requiring no φ dependence and insisting that the incoming wave be formally equal to $i\delta(1-\cos\theta)(kr)^{-1}\,e^{-ikr}$, as would be true in ordinary scattering theory. His solution is completely equivalent to (3.18) and its generalization in the Appendix. One might be disturbed about the argument of Sec. II about φ dependence, but a little thought shows that the analogue to (2.4) in the presence of a vector potential is

$$\mathbf{b} = [\nabla_k\alpha']_\perp,$$
$$\alpha' = \alpha - (e/\hbar c)\int^{\mathbf{x}} \mathbf{A}\cdot d\mathbf{x}. \tag{3.28}$$

This yields

$$\alpha' \sim 0, \quad z \to +\infty$$
$$\alpha' \sim -(2eg/\hbar c)\varphi, \quad z \to -\infty \tag{3.29}$$
$$\hat{k} = -\hat{z}.$$

The relation (3.28) in fact both justifies and generalizes Banderet's requirement of no φ dependence in his solution. The natural requirement is that α' be independent of φ for the incident wave. For a wave traveling in the *positive* z direction, this implies $\alpha \sim (2eg/\hbar c)\varphi$, so that one obtains

$$\alpha' \sim 0, \quad z \to -\infty,$$
$$\alpha' \sim +(2eg/\hbar c)\varphi, \quad z \to +\infty, \tag{3.29'}$$
$$\hat{k} = +\hat{z}.$$

To recapitulate the results of this section, the nonrelativistic quantum problem of charge-monopole scattering may be solved consistently with the requirements of Sec. II. Two approaches may be used, the Dirac method and the spin method. The two methods are completely equivalent. The Dirac technique pays the price of introducing an interaction which appears to be singular along a half-line, but in return it avoids the redundancy of the $(2s+1)$-component spin formalism. The Dirac wave function is obtained by a rotation of the spin wave function to radius-fixed axes.

IV. RELATIVISTIC PROBLEMS AND CONCLUSIONS

Unfortunately, there is no present relativistic theory of charges and monopoles (i.e., including antiparticle annihilation reactions).[16] However, the results of Sec. II imply conditions on such a theory which illuminate difficulties in recent attempts to construct one.

In order to see this, we must pay attention to the spin in the initial and final states in the reaction

$$e+g \to e+g. \tag{4.1}$$

First of all, could this spin belong to a new particle? Clearly it could not. Aside from the fact that no energy or momentum is lost by the two charges, the requirements of unitarity forbid one to consider the spin as an independent degree of freedom. We have already seen that, at least at small angles, the scattering is dominated by the off-diagonal spin matrix element $\langle -s|T|s\rangle$. If

[15] These solutions were obtained by I. Tamm, Z. Physik **71**, 141 (1931) and by M. Fierz, Helv. Phys. Acta **17**, 27 (1944). The radial equation is easily solved. The angular equation is best solved by recognizing that it is a special form of the differential equation for rotation functions. See, e.g., A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957), p. 65. The earlier authors, following Dirac, tacitly assumed $eg=-|eg|$. I assume the opposite.

[16] Dirac (1948, Ref. 1), and N. Cabibbo and E. Ferrari [Nuovo Cimento **23**, 1147 (1962)], have constructed field-theoretic equations of motion, but there has been no progress in obtaining even approximate solutions. Banderet (Ref. 12) discusses the relativistic problem in the sense of using relativistic kinematics for the Dirac electron.

the spin were an independent variable, then the optical theorem would fail because the spin-diagonal forward scattering amplitude would be negligible compared to the infinite total cross section. Because of the infinity, the optical theorem here only has meaning in terms of a limiting process for the unitarity relation at nonforward angles. However, this does not alter the conclusion that the unitarity relation applies to the amplitude with φ dependence factored out, and thus that the extra spin cannot belong to a new particle. The implication of this argument is that the azimuthal dependence of the amplitude is the only consequence of its spin structure, which in turn must be considered as purely kinematic, and not a reflection of additional physical variables in the problem. The spin may be defined in the center-of-mass frame for the reaction, with the standard transformations used to describe it in other Lorentz frames.

The usual S-matrix theory[17] for particle reactions has several important postulates, including Lorentz invariance, unitarity, analytic dependence on the invariant variables of the reaction, and crossing symmetry. The last of these states that the amplitude for reaction (4.1) may be analytically continued to give the amplitude for the crossed reaction

$$e+\bar{e}\rightarrow g+\bar{g}. \qquad (4.2)$$

In conventional problems, for which all spins are associated with specific particles, there are well-known procedures for relating the direct to the crossed amplitude. However, reaction (4.1) involves an extra spin which one would not expect to appear in (4.2). Thus, the crossing relation will not take a simple form, if it holds at all for this case, because the kinematic structure is different for direct and crossed reactions: "Naive" crossing symmetry must fail for charge-monopole interactions.

This result is consistent with other considerations. In a theory with a conserved parity operator P, the monopole g must reverse charge under P.[18] If g has no other internal quantum numbers, then P on g is like PC on other particles, where C is particle-antiparticle conjugation. It is easy to see that this definition implies that there is no contribution from one-photon intermediate states to reaction (4.2). Thus, although there is a pole at zero momentum transfer in the direct channel, there is no pole at zero mass in the crossed channel. Once again we have a sign that crossing symmetry fails for this reaction. For monopoles (or charges) with spin, the $1-\gamma$ contribution might be nonzero, but it is peculiar that the crossed-channel pole should depend on the particle spins, while the direct channel pole (determined by the correspondence principle) does not.

[17] G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961).

[18] This is seen most easily by requiring that the Lorentz force law (3.1) remain unchanged by parity inversion. See L. I. Schiff, Am. J. Phys. **32**, 812 (1964).

Recently, Zwanziger[19] and Weinberg,[20] in somewhat similar ways, attempted to build a monopole theory, at least for the small-angle scattering. Zwanziger found an amplitude with unacceptable analyticity properties. Weinberg, with a more restrictive definition of his theory, found an amplitude which was not Lorentz-invariant. Both theories seem to incorporate crossing symmetry. From the point of view of this work, the inconsistency of these theories could not have been predicted, but their incompatibility with small-angle charge-monopole scattering was assured by the implicit incorporation of crossing symmetry. Indeed, the amplitudes obtained do not even give the differential cross section correctly.

In conclusion, we have seen that our present knowledge of monopole theory may be derived from the correspondence principle and rotational invariance in S-matrix theory. One finds that the Dirac quantization condition holds for the renormalized or physical charges. Also, among the standard postulates of relativistic S-matrix theory, at least that of crossing symmetry must be modified for monopoles. Thus, if monopoles are found, at least a small revolution in present theoretical methods will be required to deal with them. If they do not obey the Dirac condition, even standard quantum-mechanical postulates will have to be questioned.

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APPENDIX

Let us find the solutions for nonrelativistic charge-monopole scattering for arbitrary integer values of $2s=2eg/\hbar c$.

The incoming wave is

$$\psi_{\rm inc}=\sum_l(2l+1)i^lj_l(kr)P_l(\cos\theta)\chi(m=s)$$
$$=\sum_l(2l+1)i^lj_l(kr)\langle l,0(\hat{r})|l,0(\hat{z})\rangle|s,s(\hat{z})\rangle, \quad (A1)$$

where the arguments $\hat{r}$ and $\hat{z}$ specify the axes of quantization of m_l or m_s, as the case may be.[21] Since $\mathbf{s}\cdot\hat{r}$ commutes with H, we may write the projection of $\psi_{\rm inc}$ on states in which the magnetic quantum number of $\mathbf{s}$ is specified in the radial direction:

$$\langle m(\hat{r})|\psi_{\rm inc}\rangle=\sum_l(2l+1)i^lj_l(kr)\times\langle l,0(\hat{r})|l,0(\hat{z})\rangle\langle s,m(\hat{r})|s,s(\hat{z})\rangle. \quad (A2)$$

If we also select states of definite J, then the angular part of the Hamiltonian will have a specific value for

[19] D. Zwanziger, Phys. Rev. **137**, B647 (1965).

[20] S. Weinberg, Phys. Rev. **138**, B988 (1965).

[21] $P_l(\cos\theta)=d_{00}{}^l(\theta)=\langle l,0(r)|l,0(z)\rangle$. See Rose (Ref. 11), pp. 52, 60.

each J:

$$\langle m|P_J|\psi_{\rm inc}\rangle=\sum_l(2l+1)i^l j_l(kr)C(J,l,s;0,m)\times C(J,l,s;0,s)\langle J,m(\hat{r})|J,s(\hat{z})\rangle, \quad (A3)$$

where the C's are vector coupling coefficients.[22]

Using[23]

$$H_J=\frac{-\hbar^2}{2\mu}\left[\frac{1}{r}\partial_r^2 r-\frac{J(J+1)-s^2}{r^2}\right]=\frac{\hbar^2k^2}{2\mu} \quad (A4)$$

we can match $\psi_{\rm inc}$ with the incoming part of eigenstates of H:

$$\langle m|P_J|\psi\rangle=\sum_l(2l+1)(-1)^l C(J,l,s;0,m)\times C(J,l,s;0,s)i^{-q}e^{i(s-m)\varphi}d_{m,s}{}^J(\theta)j_q(kr), \quad (A5)$$

$$q+\tfrac{1}{2}=[(J+\tfrac{1}{2})^2-s^2]^{1/2},$$

where we have used the standard definition of the rotation functions or helicity amplitudes $d_{m,s}{}^J(\theta)$.[24] With the help of the sum rule[25]

$$\sum_l(2l+1)C(Jls;s0)C(Jls;m0)(-1)^l=(2J+1)(-1)^{J-s}\delta_{m,-s}, \quad (A6)$$

we finally obtain

$$\langle m(\hat{r})|\psi\rangle=e^{2is\varphi}\delta_{m,-s}\times\sum_J(2J+1)(-1)^{J-s}i^{-q}j_q(kr)d_{-s,s}{}^J(\theta), \quad (A7)$$

which is equivalent to the Banderet[12] solution, and yields the correct small-angle behavior:

$$\langle -s(\hat{z})|f|s(\hat{z})\rangle \underset{\theta\ll 1}{\approx} e^{2is\varphi}[(-1)^{-s}/2ik]\times\sum_J(2J+1)d_{-s,s}{}^J(\theta)=i(-1)^s e^{2i(eg/\hbar c)\varphi}\frac{eg(v/c)}{2\mu v^2\sin^2(\theta/2)}. \quad (A8)$$

The last expression is derived, following Banderet, by recognition of the coefficients in the expansion

$$(1-\cos\theta)^{-1}=\tfrac{1}{2}\sum_J(2J+1)C_J d_{-s,s}{}^J(\theta),$$

$$C_J=\int_{-1}^{1}d\cos\theta(1-\cos\theta)^{-1}d_{-s,s}{}^J=(s)^{-1}(-1)^{2s+1}. \quad (A9)$$

The integral C_J is obtained from successive integration by parts with the help of the Rodrigues formula,[26]

$$d_{-s,s}{}^J(\cos^{-1}x)=\frac{(-1)^{J+s}d^{J-s}(1-x)^{2s}(1-x^2)^{J-s}/dx^{J-s}}{2^J(J-s)!(1-x)^s}. \quad (A10)$$

Of course, the caveat in the text about the interpretation of the asymptotic scattered wave applies also to (A8). The required condition is $kr\theta^2\gg s$.

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[22] E. P. Wigner, *Group Theory* (Academic Press Inc., New York, 1959), Chap. 17.

[23] This is simply the quantum version of (3.7). The use of s^2 instead of $s(s+1)$ maintains agreement with the Dirac Hamiltonian. The distinction does not affect small-angle scattering, but it does maintain the positive definite character of H for $J=s$. This is the lowest value of J consistent with the boundary condition $\mathbf{s}\cdot\hat{r}=-s$, since $\mathbf{s}\cdot\hat{r}=\mathbf{J}\cdot\hat{r}$.

[24] Wigner (Ref. 22), Chap. 15; Edmonds (Ref. 15), Chap. 4; Rose (Ref. 11), Chap. 4; Jacob and Wick (Ref. 10).

[25] Rose (Ref. 11), pp. 41–42.

[26] Edmonds (Ref. 15), p. 58.