Thomson's monopoles

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(Received 21 November 1975; revised 9 February 1976)

The angular momentum **L** of the electromagnetic field due to an electric point charge e, and a magnetic point charge (pole) g, is calculated by several methods to obtain J. J. Thomson's result that $L = eg/c$, and **L** is directed along the line joining the electric monopole to the magnetic monopole. The relation to Dirac's monopoles is discussed, and particle size is considered.

I. INTRODUCTION

In 1931 Dirac[1] considered the phase of the wave function describing the motion of an electron of charge $-e$ in the field of an isolated magnetic pole of strength g, and concluded that if free magnetic poles occur in nature, the electric and magnetic charges must be quantized according to the product relation

$$eg = \tfrac{1}{2} n\hbar c, \tag{1.1}$$

where n is an interger, $\hbar$ is Planck's constant divided by 2π, and c is the speed of light.[2] This means that if the electron charge is the smallest electric charge, the smallest magnetic charge (pole strength) is $(137/2)\,e$.

A system consisting of an electric charge and a magnetic pole was considered by Thomson.[3,4] Figure 1 shows an electric point charge e at A and a magnetic pole g at B. Thomson obtained, without showing details, that the angular momentum of the electromagnetic field of the system is in the direction AB and equals eg/c. By invoking the quantization rule for angular momentum, Saha[5] and Wilson[6] set $eg/c = n\hbar/2$, and obtained very simply Dirac's result. A good discussion on magnetic poles is now available in a text by Jackson.[7]

In this paper several derivations of Thomson's result are presented with the hope that the methods discussed would be of interest to students and teachers of electricity and magnetism and would find applications in other problems. The system considered, Thomson's monopoles, is of interest since it is relevant to the intensive research efforts on magnetic poles,[8–10] and since it offers, as Thomson[4] showed, a simple example illustrating the conservation of the total angular momentum of the system which consists of the angular momentum of the field set up by the charges and the orbital angular momentum of the charges themselves.

Three different derivations are given: Sec. II deals with a conventional derivation using polar coordinates. This method, which is very likely Thomson's unpublished method, must be known, in one form or another, to many readers. It is included for completeness and to introduce the other two methods. Section III uses the results of potential theory and the integrations are carried out by inspection. Section IV discusses the problem in its natural coordinate system, the prolate spheroidal system, which exhibits the symmetry of the problem. Finally, a simple derivation is given in the Appendix by a slight modification of Thomson's torque argument.[4] Gaussian units are used and the result in mks units is given in the Appendix.

II. POLAR COORDINATES

Figure 1 shows the geometry of the model. The z axis is in the direction AB and the spacing of the charges is a. The electric and magnetic fields at P are $\mathbf{E} = e\mathbf{r}_1 r_1^{-3}$, and $\mathbf{B} = g\mathbf{r}_2 r_2^{-3}$. The linear momentum density of the field is defined by $\mathbf{E} \times \mathbf{B}/(4\pi c)$, and hence the momentum element at P is

$$d\mathbf{P} = eg(4\pi c)^{-1}(r_1 r_2)^{-2} \sin\theta_{12}\, d\tau\, \hat{e}_\phi, \tag{2.1}$$

where $d\tau$ is a volume element, and $\hat{e}_\phi$ is a unit vector in the ϕ direction (out of the plane of the paper). We observe that the momenta elements at P and P' (the image of P in the z axis) form a couple whose axis is the z axis, and the system is analogous to a spinning top.[3] Thus, *the total electromagnetic momentum of the system is zero, and the total angular momentum (moment of momentum) is directed along the z axis, and can be obtained by taking the moment of the momentum elements about any point in space.* By taking the z component of the moment of $d\mathbf{P}$ about any point on the z axis we obtain the element of angular momentum

$$dL_z = eg(4\pi c)^{-1}(r_1 r_2)^{-2} \rho \sin\theta_{12}\, d\tau. \tag{2.2}$$

Take the origin at A and substitute

$$d\tau = 2\pi r_1^2 \sin\theta_1\, d\theta_1\, dr_1,$$

$$\rho = r_1 \sin\theta_1, \quad \sin\theta_{12}/a = \sin\theta_1/r_2$$

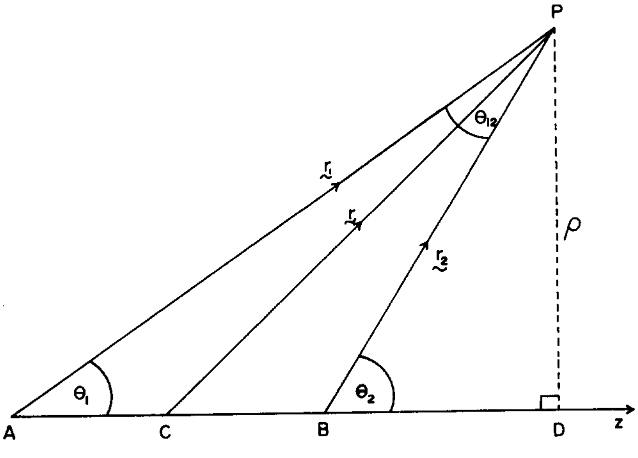

Fig. 1. Geometry of the model.

to obtain

$$L_z=\frac{ega}{2c}\int\frac{r_1\sin^3\theta_1}{r_2^3}\,d\theta_1\,dr_1. \qquad (2.3)$$

With the substitutions $r_1=as$, $\theta_1=\theta$ (dummy variable), and $r_2^2=a^2(1+s^2-2s\cos\theta)$ we obtain

$$L_z=\frac{eg}{2c}\int\frac{s\sin^3\theta\,d\theta\,ds}{(1+s^2-2s\cos\theta)^{3/2}}, \qquad (2.4)$$

which shows that L_z is independent of the distance a between the charges.[11] Integrate by parts over θ to obtain

$$L_z=\frac{eg}{c}\int_0^\infty ds\int_0^\pi\frac{\cos\theta\sin\theta\,d\theta}{(1+s^2-2s\cos\theta)^{1/2}}. \qquad (2.5)$$

Expand $(1+s^2-2s\cos\theta)^{-1/2}$ in spherical harmonics and integrate the only surviving term $\cos^2\theta$ to obtain[12]

$$\int_0^\pi\frac{\sin\theta\cos\theta\,d\theta}{(1+s^2-2s\cos\theta)^{1/2}}=\begin{cases}\frac{2}{3}s, & s<1,\\ \frac{2}{3}s^{-2}, & s>1,\end{cases} \qquad (2.6)$$

which together with (2.5) gives Thomson's result

$$L_z=eg/c. \qquad (2.7)$$

Alternatively, the integral in (2.5) can be written

$$J\equiv\int_0^\infty ds\int_0^\pi\frac{\cos\theta\sin\theta\,d\theta}{(1+s^2-2s\cos\theta)^{1/2}}$$

$$=\int\frac{1}{(1+s^2-2s\cos\theta)^{1/2}}\frac{\cos\theta}{2\pi s^2}\,d^3s, \qquad (2.8)$$

which by comparison with the formula giving the potential $\psi(\mathbf{r})$ for a given charge distribution $\rho(\mathbf{r})$, namely

$$\psi(\mathbf{r})=\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')\,d^3r', \qquad (2.9)$$

shows that J is the potential at the point $z=1$ on the polar axis (location of the magnetic pole) for the charge distribution $\cos\theta/(2\pi s^2)$ surrounding the origin (location of the electric charge). But the solution of Poisson's equation

$$\Delta\psi=-2\cos\theta/r^2 \qquad (2.10)$$

is simply[13]

$$\psi(r)=\cos\theta. \qquad (2.11)$$

Hence, $J=\psi(\theta=0)=1$.

We shall now exploit this result of potential theory and Gauss's theorem to construct a simple derivation.

III. POTENTIAL THEORY SOLUTION

Take the angular momentum about the point C the center of AB. Since

$$\mathbf{r}\times(\mathbf{E}\times\mathbf{B})=\mathbf{r}\cdot\mathbf{B}\mathbf{E}-\mathbf{r}\cdot\mathbf{E}\mathbf{B},$$

and $2\mathbf{r}=\mathbf{r}_1+\mathbf{r}_2$ and $\mathbf{a}=\mathbf{r}_1-\mathbf{r}_2$, we have

$$d\mathbf{L}=\frac{eg}{8\pi c}\left(\frac{\mathbf{r}_1}{r_2r_1^3}-\frac{\mathbf{r}_2}{r_1r_2^3}+\frac{\mathbf{r}_1\cdot\mathbf{r}_2}{r_1^3r_2^3}\mathbf{a}\right)d\tau, \qquad (3.1)$$

$$dL_z=\frac{eg}{8\pi c}\left(\frac{\cos\theta_1}{r_2r_1^2}-\frac{\cos\theta_2}{r_1r_2^2}+a\frac{\mathbf{r}_1\cdot\mathbf{r}_2}{r_1^3r_2^3}\right)d\tau. \qquad (3.2)$$

By (2.8), (2.10), and (2.11) the integrals of the first terms in (3.2) are

$$\int\frac{\cos\theta_1}{2\pi r_2r_1^2}\,d^3r_1=\int\frac{-\cos\theta_2}{2\pi r_1r_2^2}\,d^3r_2=1; \qquad (3.3)$$

the first integral is the potential at B due to the charge distribution $\cos\theta_1/(2\pi r_1^2)$ centered at A, and the second integral is minus the potential at A due to a similar charge distribution at B; in (2.11), $\theta=0$ for the first integral and $\theta=\pi$ for the second integral.

In the last term in (3.2) write $d\tau=d^3r_2=dS\,dr_2$, where S is the surface of a sphere of center B and radius r_2 to obtain

$$\frac{ea}{4\pi}\int\frac{\mathbf{r}_1\cdot\mathbf{r}_2}{r_1^3r_2^3}\,d^3r_2=\frac{a}{4\pi}\int\frac{dr_2}{r_2^2}\int\mathbf{E}\cdot d\mathbf{S}$$

$$=ea\int_0^\infty\frac{dr_2}{r_2^2}\,U(r_2-a)=e, \qquad (3.4)$$

where the surface integral is evaluated by Gauss's theorem, namely, the electric flux across the sphere of radius r_2 is $4\pi e$ if $r_2>a$, and zero otherwise, or flux $=4\pi eU(r_2-a)$, where U is the step function. The result $L_z=eg/c$ follows from Eqs. (3.2)–(3.4).

IV. PROLATE SPHEROIDAL COORDINATES

The coordinates of P are (u,v,ϕ), where ϕ is the azimuthal angle, and the constant u and v surfaces are confocal prolate ellipsoids and hyperboloids, respectively, with A and B as foci. With C as origin and $\rho^2=x^2+y^2$, the surfaces are defined by[14]

$$u=r_1+r_2,\quad a<u<\infty, \qquad (4.1)$$

$$v=r_2-r_1,\quad -a<v<a, \qquad (4.2)$$

or, equivalently by geometry,

$$\frac{z^2}{u^2}+\frac{\rho^2}{u^2-a^2}=\frac{1}{4}, \qquad (4.3)$$

$$\frac{z^2}{v^2}-\frac{\rho^2}{a^2-v^2}=\frac{1}{4}. \qquad (4.4)$$

To obtain L_z in this coordinate system we need to express dL_z of (2.2) in terms of u and v. Evidently, $r_1=(u-v)/2$, $r_2=(u+v)/2$, $\nabla u=\hat{r}_1+\hat{r}_2$, where the carat denotes a unit vector, $\nabla v=\hat{r}_2-\hat{r}_1$, $\nabla u\cdot\nabla v=0$ (verifying the orthogonality of the ellipsoids and hyperboloids), $|\nabla u|^2=2(1+\cos\theta_{12})$, and $|\nabla v|^2=2(1-\cos\theta_{12})$. Hence, the volume element which is $2\pi\rho\,du\,dy/(|\nabla u|\,|\nabla v|)=(\pi\rho/\sin\theta_{12})\,du\,dv$. By eliminating z^2 from (4.3) and (4.4) we obtain

$$\rho^2=(a^2-v^2)(u^2-a^2)/(4a^2). \qquad (4.5)$$

By performing the proper substitutions in (2.2) we obtain

$$L_z=\frac{eg}{c}\int_{-a}^{a} dv\int_{a}^{\infty} \frac{(a^2-v^2)(u^2-a^2)}{a^2(u^2-v^2)^2}\,du. \quad (4.6)$$

Now, let $u=a/x$, and $v=ay$ to obtain the symmetric integral

$$L_z=\frac{2eg}{c}\int_0^1\int_0^1 \frac{(1-x^2)(1-y^2)}{(1-x^2y^2)^2}\,dx\,dy, \quad (4.7)$$

where the factor 2 accounts for limiting the y integration to positive values. It remains to show that the integral is 1/2. Expand $(1-x^2y^2)^{-2}$ in power series and integrate term by term. Thus,

$$\int_0^1\int_0^1 [1-(x^2+y^2)+x^2y^2]\sum_{n=0}^{\infty}(n+1)x^{2n}y^{2n}\,dx\,dy=\sum_n (n+1)\left(\frac{1}{(2n+1)^2}-\frac{2}{(2n+1)(2n+3)}+\frac{1}{(2n+3)^2}\right)$$

$$=\sum_n (n+1)\frac{4}{(2n+1)^2(2n+3)^2}=\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}-\frac{1}{(2n+3)^2}=\frac{1}{2}\left(1+\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}-\sum_{n=0}^{\infty}\frac{1}{(2n+3)^2}\right)=\frac{1}{2}, \quad (4.8)$$

which is the desired result.[15]

V. PARTICLE SIZE[16]

Since a finite electrostatic (magnetostatic) energy requires a charge of finite size, let us calculate **L** for charges of finite size. First, assume in Fig. 1 the electric charge to be a sphere of radius R, and the magnetic charge to be a point charge. For spherical charge distribution, the electric field is radial and is of magnitude $E(r)$. By the methods leading to Eq. (2.6),

$$L_z=(2g/3c)\int r^2E(r)f(r)\,dr, \quad (5.1)$$

$$f(r)=r/a^2, \quad r<a,$$
$$=a/r^2, \quad r>a. \quad (5.2)$$

If the electric charge is *distributed uniformly over the volume of the sphere R*, the above integration gives

$$L_z=(eg/c)(1-R^2/5a^2), \quad a>R,$$
$$=(ega/5cR^3)(5R^2-a^2), \quad a<R. \quad (5.3)$$

If, on the other hand, the electric charge is *distributed uniformly over the surface of the sphere R*, we have

$$L_z=(eg/c)(1-R^2/3a^2), \quad a>R,$$
$$=2ega/3cR, \quad a<R. \quad (5.4)$$

If both charges are now rigid (impenetrable) spheres of radii R_e and R_g, Eqs. (5.3) and (5.4) give

$$L_z=(eg/c)[1-(R_e{}^2+R_g{}^2)/\lambda a^2], \quad (5.5)$$

where $\lambda=5$ for uniform volume charge distributions, and $\lambda=3$ for uniform surface charge distributions. Thus, if the charges remain a few diameters away from each other, their field angular momentum is practically that of point charges. By considering charge distributions concentrated near the center of the sphere, the size effect can be made negligible.

There is a well-known difficulty with the conservation of the angular momentum of point charges: In a head-on collision, the field angular momentum **L** is reversed if one charge goes through the other, and since the orbital angular momentum **l** remains zero, the total angular momentum is not conserved.[17] By considering classically the monopoles as rigid smooth spheres with charge distributions concentrated near the centers, this difficulty is avoided.

ACKNOWLEDGMENTS

It is a pleasure to thank my teacher Professor P. Morrison for introducing me to this problem in his course on electrodynamics in 1954, and to thank Professor C. Kittel for reviving my interest in the problem and for valuable advice and encouragement.

APPENDIX

In Fig. 1 let the x axis be in the direction DP. Allow the magnetic pole B to move relative to A with a nonrelativistic velocity **v** in the x direction for an infinitesimal time δt. The magnetic pole experiences a Lorentz force of magnitude $egv/(a^2c)$ in the positive y direction which is equal and opposite to the Lorentz force exerted by the pole on the electric charge. The torque produces a change in the orbital angular momentum of the charges $\delta\mathbf{l}$ which points in the negative x direction and is given by $\delta l_x=egv\delta t/(ac)$. The electromagnetic field angular momentum **L** is along AB, and since AB rotates by the angle $v\delta t/a$, the change in the field angular momentum $\delta\mathbf{L}$ is in the x direction and $\delta L_x=Lv\delta t/a$. Thomson[4] inserts $L=eg/c$ to prove $\delta(\mathbf{l}+\mathbf{L})=0$. Conversely, if we assume the conservation of angular momentum we obtain $L=eg/c$. Alternatively, if we apply an external torque to the charges to balance the Lorentz torque, we have $\delta l_x=0$ and $\delta L_x=egv\delta t/(ac)$ or $L=eg/c$.

In mks units the momentum density is defined by $\mathbf{E}\times\mathbf{H}/c^2$ or $\mathbf{D}\times\mathbf{B}$ since $\epsilon_0\mu_0=c^{-2}$. This multiplies the cgs result for L_z by the factor $\mu_0c/(4\pi)$ and Dirac's result assumes the form $L_z=eg\mu_0/(4\pi)=n\hbar/2$ which gives 3.3×10^{-9} A m for the smallest g.

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[11]The integral in (2.4) can be performed over s first by the substitution $s = \cos\theta + \sin\theta \tan\phi$ followed by an elementary integration over θ to give (2.7).
[12]The result (2.6) can be obtained by the substitution $u^2 = 1 + s^2 - 2s \cos\theta$.
[13]Note $\Delta \cos\theta = -(L^2/r^2)\cos\theta = -(2/r^2)\cos\theta$, where L^2 is the square of the angular momentum operator.
[14]For a general discussion on ellipsoidal coordinates see, e.g., H. Bateman, *Partial Differential Equations* (Dover, New York, 1944), Chap. 8.
[15]The integral can also be performed routinely over x by the substitution $x = \sin\theta/y$. The resulting y integration

$$\frac{1}{2}\int_0^1 \left(\frac{1-y^4}{2y^3} \ln\frac{1+y}{1-y} - \frac{1-y^2}{y^2}\right) dy,$$

can be performed either by successive partial integrations with proper care when taking limits, or by expanding the integrand in a power series and arriving at the last line of (4.8).
[16]This section was added in proof.
[17]I am indebted to Professor Kittel for bringing this to my attention. This problem is avoided in quantum mechanics since the wave function for the relative motion of the monopoles vanishes at the origin. See, e.g., H. J. Lipkin, W. I. Weisberger, and M. Peshkin, Ann. Phys. **53**, 203 (1969).